# Eigenfunction for dissipative dynamic operators and the attractor of the dissipative structure

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This study shows that the states with the minimum dissipation rate in general dissipative dynamic systems are expressed by the eigenfunctions for the dissipative dynamic operators. These eigenfunctions are shown to constitute the self-organized and self-similar decay phase as the attractor of the dissipative structure. A typical example applied to incompressible viscous fluid is presented to describe a physical picture of self-organization and bifurcation of the dissipative structure.

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Dissipative structures realized in dissipating nonlinear dynamical systems have attracted much attention in many research fields. They include various self-organized structures in thermodynamic systems [1, 2], the force-free fields of cosmic magnetism [3], the self-organized relaxed state in the magnetized fusion plasmas [4–7], and the helical flow patterns which follow turbulent puffs [8]. We can see some common mathematical structure among the self-organized relaxed states of the dissipative structure and also among the proposed theories themselves to explain those dissipative structures [9, 10]. The study of the common universal mathematical structures embeded in dissipative nonlinear dynamic systems and leading to those dissipative structures is an area of deep interest. In this paper, we investigate the mathematical structure and propose a theory which leads to the attractor of the dissipative structure. A typical application of the theory is also shown.

Quantities with $n$ elements in dynamic systems of interest shall be expressed as $\mathbf{q}(t,\mathbf{x}) = (q_1(t,\mathbf{x}), q_2(t,\mathbf{x}), \ldots, q_n(t,\mathbf{x}))$. We first consider an ideal nondissipative dynamic system which may be described by

$$\frac{\partial q_i}{\partial t} = L_i^N[\mathbf{q}] \,, \tag{1}$$

where $L_i^N[\mathbf{q}]$ are nondissipative, linear or nonlinear dynamic operators, such as $q_i = \mathbf{u}$ and $L_i^N[\mathbf{q}] = -\boldsymbol{\nabla} p/\rho - \boldsymbol{\nabla} u^2/2 + \mathbf{u}\times\boldsymbol{\omega}$ in ideal incompressible fluid dynamics. When the dynamic system is in an unstable state, the dynamic operator $L_i^N[\mathbf{q}]$ may lead to the rapid growth of perturbations and/or to turbulent phases. In some cases, a nonlinearity of $L_i^N[\mathbf{q}]$ may lead to nonlinear saturation of perturbations. When no external input is applied, global autocorrelations $W_{ii} = \int q_i q_i \, dV = \int q_i^2 \, dV$ across the space volume of the system, which usually represent the system's total energy, are conserved because there is no dissipation by the nondissipative operator $L_i^N[\mathbf{q}]$. We accordingly obtain

$$\frac{\partial W_{ii}}{\partial t} = 2\int q_i \frac{\partial q_i}{\partial t}\, dV = 2\int q_i L_i^N[\mathbf{q}]\, dV = 0 \,. \tag{2}$$

In other words, Eq. (2) indicates that the definition of the nondissipative dynamic operator $L_i^N[\mathbf{q}]$ is such that $W_{ii}$ is the time invariant of the dynamic system. We can treat the steady state of $\partial q_i/\partial t = 0$ to obtain the equilibrium equations from Eq. (1) as follows:

$$L_i^N[\mathbf{q}] = 0 \,. \tag{3}$$

We cannot usually find any spatial profiles peculiar or unique to $q_i$ from the equilibrium equations, Eq. (3), themselves.

We now proceed to a dissipative dynamic system which may be described by

$$\frac{\partial q_i}{\partial t} = L_i^N[\mathbf{q}] + L_i^D[\mathbf{q}] \,, \tag{4}$$

where $L_i^N[\mathbf{q}]$ are the nondissipative dynamic operators shown above, and $L_i^D[\mathbf{q}]$ are dissipative, linear or nonlinear dynamic operators, such as $q_i = \mathbf{u}$ and $L_i^D[\mathbf{q}] = (\nu/\rho)\nabla^2\mathbf{u}$ in the Navier-Stokes equation for incompressible viscous fluid dynamics with the coefficient of viscosity $\nu$. When the dynamic system has some unstable regions, the nondissipative dynamic operator $L_i^N[\mathbf{q}]$ may become dominant and lead to the rapid growth of perturbations there and further to turbulent phases. This may yield spectrum transfers toward the larger-wave-number region in $q_i$ distributions, as in the normal energy cascade demonstrated by three-dimensional magnetohydrodynamic (MHD) simulations in [11–13] or in the turbulent region in the turbulent puff shown in [8]. When the larger wave number becomes a large fraction of the spectrum, the dissipative dynamic operator $L_i^D[\mathbf{q}]$ may become dominant to yield higher dissipations for the larger-wave-number components of $W_{ii}$. In this rapid dissipation phase, which is far from equilibrium, the unstable regions in the dynamic system are considered to vanish to produce a stable configuration again. Certain nonlinear processes are assumed implicitly here as some of the dominant processes during this transition from the unstable to the stable state. When no external input is applied, the steady state of $\partial q_i/\partial t = 0$ and $\partial W_{ii}/\partial t = 0$ can no longer be realized because of dissipation. However, we can treat a quasisteady state where the equilibrium equations, Eq. (3), are applicable approximately. In this quasisteady state with Eq. (3), the dissipation rate of $W_{ii}$ is written as follows, using Eqs. (2)–(4):

$$\frac{\partial W_{ii}}{\partial t} = 2\int q_i \frac{\partial q_i}{\partial t}\, dV = 2\int q_i L_i^D[\mathbf{q}]\, dV\,. \qquad (5)$$

We then investigate whether the quasisteady state with the minimum dissipation rate of $|\,\partial W_{ii}/\partial t\,|$ after the turbulent and nonlinear relaxation phase has some unique spatial profiles of $q_i$, similarly to [3, 9]. This is a typical problem of variational calculus with respect to the spatial variable $\mathbf{x}$ to find the spatial profiles of $q_i$ that satisfy the following:

$$\min \left| \frac{\partial W_{ii}}{\partial t} \right| \text{ for a given value of } W_{ii}. \qquad (6)$$

We use the notation $\mathbf{q}^*(W_{ii}, \mathbf{x})$ or simply $q_i^*$ for the profiles of $q_i$ that satisfy Eq. (6). Since $\partial W_{ii}/\partial t$ usually has a negative value, the mathematical expressions for Eq. (6) are written as follows, defining a functional $F$ with use of a Lagrangian multiplier $\alpha$:

$$F \equiv -\frac{\partial W_{ii}}{\partial t} - \alpha W_{ii}\,, \qquad (7)$$

$$\delta F = 0\,, \qquad (8)$$

$$\delta^2 F > 0, \qquad (9)$$

where $\delta F$ and $\delta^2 F$ are the first and the second variations of $F$ with respect to the variation $\delta\mathbf{q}(\mathbf{x})$ only for the spatial variable $\mathbf{x}$. Substituting Eq. (5) into Eqs. (7) and (8), we obtain

$$\delta F = -2\int \{\, \delta q_i (L_i^D[\mathbf{q}] + \alpha q_i) + q_i \delta L_i^D[\mathbf{q}]\,\}\, dV = 0\,. \qquad (10)$$

We now impose the following self-adjoint property upon the dissipative dynamic operators $L_i^D[\mathbf{q}]$ :

$$\int q_i\, \delta L_i^D[\mathbf{q}] dV = \int \delta q_i L_i^D[\mathbf{q}] dV + \oint \mathbf{P}\cdot d\mathbf{S}\,, \qquad (11)$$

where $\oint \mathbf{P}\cdot d\mathbf{S}$ denotes the surface integral term which comes out as from the Gaussian theorem. The self-adjoint property of Eq. (11) is satisfied by dissipative dynamic operators such as $(\nu/\rho)\boldsymbol{\nabla}^2\mathbf{u}$ in the Navier-Stokes equation and the Ohm loss term of $-\boldsymbol{\nabla}\times(\eta\mathbf{j})$ in the magnetic field equation of the resistive MHD plasma. The surface integral term of $\oint \mathbf{P}\cdot d\mathbf{S}$ sometimes vanishes because of the boundary condition, as in the case of the ideally conducting wall. Using the self-adjoint property of Eq. (11), we obtain the following from Eq. (10):

$$\delta F = -2\int \delta q_i (2L_i^D[\mathbf{q}] + \alpha q_i)\, dV - 2\oint \mathbf{P}\cdot d\mathbf{S} = 0\,. \qquad (12)$$

We then obtain the Euler-Lagrange equations from the volume integral term in Eq. (12) for arbitrary variations of $\delta q_i$ as follows:

$$L_i^D[\mathbf{q}^*] = -\frac{\alpha}{2}\, q_i^*\,. \qquad (13)$$

We find from Eq. (13) that the profiles of $q_i^*$ are given by the eigenfunctions for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$, and therefore have a feature uniquely determined by the operators $L_i^D[\mathbf{q}^*]$ themselves. As a boundary value problem, we may assume that Eq. (13) can be solved for given boundary values of $q_i$. The value of the Lagrangian multiplier $\alpha$ is determined by using the given value of $W_{ii}$ for the global constraint, as is common practice in the variational calculus. Since we cannot *a priori* predict the value of $W_{ii}$ at the state with the minimum dissipation rate for every dissipative dynamic system, we have to measure the value of $W_{ii}$ at such a state in order to determine the value of $\alpha$. However, we can predict the type of the profile $q_i^*$ for every dissipative dynamic system by using Eq. (13), if the operator $L_i^D[\mathbf{q}^*]$ is given. Substituting the eigenfunctions of Eq. (13) and the approximate equilibrium equation of Eq. (3) for the quasisteady state into Eq. (4), we obtain the following:

$$\frac{\partial q_i^*}{\partial t} \cong -\frac{\alpha}{2}\, q_i^*\,, \qquad (14)$$

$$q_i^* \cong q_{iR}^*(\mathbf{x})\, e^{-(\alpha/2)t}\,, \qquad (15)$$

$$W_{ii}^* = \int (q_i^*)^2\, dV \cong e^{-\alpha t}\int (q_{iR}^*)^2\, dV\,, \qquad (16)$$

$$\frac{\partial W_{ii}^*}{\partial t} \cong -\alpha\, W_{ii}^*\,, \qquad (17)$$

where $q_{iR}^*(\mathbf{x})$ denotes the eigensolution for Eq. (13) which is supposed to be realized at the state with the minimum dissipation rate during the time evolution of the dynamical system of interest. We find from Eq. (15) that the eigenfunctions $q_i^*$ for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$ constitute the self-organized and self-similar decay phase during the time evolution of the present dynamic system. We see from Eq. (17) that the factor $\alpha$ of Eq. (13), the Lagrangian multiplier, is equal to the decay constant of $W_{ii}$ at the self-organized and self-similar decay phase. Since the present dynamic system evolves basically by Eq. (4), the dissipation by $L_i^D[\mathbf{q}^*]$ of Eq. (13) during the self-similar decay couples with $L_i^N[\mathbf{q}]$ and the boundary wall conditions to cause gradual deviation from self-similar decay. This gradual deviation may yield some new unstable region in the dynamic system. When some external input is applied in order to recover the dissipation of $W_{ii}$, the present dynamic system is considered to return repeatedly close to the self-organized and self-similar decay phase. Observation of the time evolution of the system of interest for long periods reveals a physical picture in which the system appears to be repeatedly attracted towards and trapped in the self-organized and self-similar decay phase of Eq. (15). The system stays in this phase for the longest time during each cycle of the time evolution because this is where it has the minimum dissipation rate. In this sense, the eigenfunctions $q_i^*$ of Eq. (13) for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$ are "the attractors of the dissipative structure" introduced by Prigogine [1, 2]. It has been reported in [9] that Eq. (13) leads to $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}^* =$

$\kappa^2\mathbf{u}^*$ in the case of incompressible viscous fluids and to $\nabla \times (\eta \mathbf{j}^*) = \alpha \mathbf{B}^*/2$ in the case of resistive MHD plasmas. Both have been shown to constitute the self-similar decay phase.

Using Eq. (9), we next discuss the mode transition point or bifurcation point of the self-organized dissipative structure. Substituting Eq. (5) into Eqs. (7) and (9), we obtain

$$\delta^2 F = -2 \int \delta q_i \left( \delta L_i^D[\mathbf{q}] + \frac{\alpha}{2}\, \delta q_i \right) dV > 0 . \quad (18)$$

We consider here the following associated eigenvalue problem for critical perturbations $\delta q_i$ that make $\delta^2 F$ in Eq. (18) vanish:

$$( \delta L_i^D[\mathbf{q}] )_k + \frac{\alpha_k}{2}\, \delta q_{ik} = 0 , \quad (19)$$

with boundary conditions given for $\delta q_i$, for example, $\delta q_i = 0$ at the boundary wall. Here $\alpha_k$ is the eigenvalue and $( \delta L_i^D[\mathbf{q}] )_k$ and $\delta q_{ik}$ denote the eigensolution. Substituting the eigensolution $\delta q_{ik}$ into Eq. (18) and using Eq. (19), we obtain the following:

$$\delta^2 F = ( \alpha_k - \alpha ) \int \delta {q_{ik}}^2 \, dV > 0 . \quad (20)$$

Since Eq. (20) is required for all eigenvalues, we obtain the following condition for the state with the minimum dissipation rate:

$$0 < \alpha < \alpha_1 , \quad (21)$$

where $\alpha_1$ is the smallest positive eigenvalue and $\alpha$ is assumed to be positive. When the value of $\alpha$ goes beyond the condition of Eq. (21), as when $\alpha_1 < \alpha$, then the mixed mode, which consists of the basic mode by the solution of Eq. (13) where $\alpha = \alpha_1$ and the lowest eigenmode of Eq. (19), becomes the self-organized dissipative structure with the minimum dissipation rate. This result for the bifurcation point of the self-organized dissipative structure has the same mathematical structure as that for the self-organized relaxed state of the resistive MHD plasmas [6, 7, 9, 10].

We now show a typical application of this theory to the incompressible viscous fluid described by the Navier-Stokes equation:

$$\rho \frac{d\mathbf{u}}{dt} = - \nabla p + \nu \nabla^2 \mathbf{u}, \quad (22)$$

where $\rho$, $\mathbf{u}$, and $p$ are the fluid mass density, the fluid velocity, and the pressure, respectively, and $\nabla \cdot \mathbf{u} = 0$. For simplicity, we assume $\nu$ to be spatially uniform. Using $\nabla \cdot \mathbf{u} = 0$ and the two vector formulas of $\nabla u^2 = 2\mathbf{u} \times (\nabla \times \mathbf{u}) + 2(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $\nabla^2 \mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u}$, Eq. (22) is rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} = - \frac{\nabla p}{\rho} - \frac{1}{2}\nabla u^2 + \mathbf{u}\times\boldsymbol{\omega} - \frac{\nu}{\rho}\nabla\times\nabla\times\mathbf{u} , \quad (23)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. We find from Eq. (23) that $L_i^N[\mathbf{q}] = - \nabla p/\rho - \nabla u^2/2 + \mathbf{u}\times\boldsymbol{\omega}$ and $L_i^D[\mathbf{q}] = - (\nu/\rho)\nabla\times\nabla\times\mathbf{u}$, where $q_i \equiv \mathbf{u}$. The equilibrium equation of (3) is now given as $\nabla p + (\rho/2)\nabla u^2 = \rho(\mathbf{u}\times\boldsymbol{\omega})$. Substituting these two operators of $L_i^N[\mathbf{q}]$ and $L_i^D[\mathbf{q}]$ into Eqs. (4)–(12), and using $\delta\boldsymbol{\omega} = \nabla\times\delta\mathbf{u}$, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$, and the Gaussian theorem, we obtain the following:

$$\begin{aligned}\delta F = {} & 2 \int \delta\mathbf{u}\cdot\left(\frac{2\nu}{\rho}\nabla\times\nabla\times\mathbf{u} - \alpha\mathbf{u}\right) dV \\ & + \frac{2\nu}{\rho}\oint [\delta\mathbf{u}\times(\nabla\times\mathbf{u}) + (\nabla\times\delta\mathbf{u})\times\mathbf{u}]\cdot d\mathbf{S} \\ = {} & 0 . \end{aligned} \quad (24)$$

Here we notice that the present dissipative operator $L_i^D[\mathbf{q}]$ satisfies the self-adjoint property of Eq. (11) as follows:

$$\begin{aligned}\int \mathbf{u}\cdot\left(\frac{\nu}{\rho}\nabla\times\nabla\times\delta\mathbf{u}\right) dV & \\ = \int \delta\mathbf{u}\cdot\left(\frac{\nu}{\rho}\nabla\times\nabla\times\mathbf{u}\right) dV & \\ + \frac{\nu}{\rho}\oint [\, \delta\mathbf{u}\times(\nabla\times\mathbf{u}) + (\nabla\times\delta\mathbf{u})\times\mathbf{u}\,]\cdot d\mathbf{S}, & \end{aligned} \quad (25)$$

where the vector formula of $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$ is used twice. We obtain the Euler-Lagrange equation from the volume integral term in Eq. (24) for the arbitrary variation $\delta\mathbf{u}$, corresponding to Eq. (13), as follows:

$$\nabla\times\nabla\times\mathbf{u}^* = \frac{\alpha\rho}{2\nu}\mathbf{u}^* . \quad (26)$$

The eigenfunction of this Eq. (26) can be obtained for a given boundary value of $\mathbf{u}$, as a boundary value problem.

Substituting the eigenfunction of Eq. (26) and the approximate equilibrium equation of $\nabla p + (\rho/2)\nabla u^2 = \rho(\mathbf{u}\times\boldsymbol{\omega})$ for the quasisteady state into Eq. (23), we obtain the following:

$$\frac{\partial \mathbf{u}^*}{\partial t} \cong - \frac{\alpha}{2}\,\mathbf{u}^* , \quad (27)$$

$$\mathbf{u}^* \cong {\mathbf{u}_R}^*(\mathbf{x})\, e^{-(\alpha/2)t} , \quad (28)$$

$$W_{ii}^* = \int (\mathbf{u}^*)^2\, dV \cong e^{-\alpha t}\int ({\mathbf{u}_R}^*)^2\, dV , \quad (29)$$

$$\frac{\partial {W_{ii}}^*}{\partial t} \cong - \alpha\, {W_{ii}}^* , \quad (30)$$

where ${\mathbf{u}_R}^*(\mathbf{x})$ denotes the eigensolution for Eq. (26) for the given boundary value of $\mathbf{u}$, which is supposed to be realized at the state with the minimum dissipation rate during the time evolution of the dynamical system of interest. We find from Eq. (28) that the eigenfunction $\mathbf{u}^*$ for the present dissipative dynamic operator $L_i^D[\mathbf{q}^*]$ constitutes the self-organized and self-similar decay phase during the time evolution of the present dynamic system. We see from Eq. (30) that the factor $\alpha$ of Eq. (26), which is the Lagrangian multiplier, is equal to the decay constant of flow energy $W_{ii}$ at the self-organized and self-similar decay phase. The eigensolution for Eq. (26) contains the helical flow solution of $\nabla \times \mathbf{u} = \kappa \mathbf{u}$, where $|\kappa| = \sqrt{\alpha\rho/2\nu}$ and the Lagrangian multiplier $\alpha$ is assumed to be positive. This helical flow solution of $\nabla \times \mathbf{u} = \kappa \mathbf{u}$ is considered to represent approximately the helical flow pattern after the turbulent puffs shown in Fig. 4 of [8].

Substituting the present dissipative dynamic operator, $L_i^D[\mathbf{q}] = -(\nu/\rho)\nabla \times \nabla \times \mathbf{u}$, into Eqs. (18)–(21), we next discuss the mode transition point or bifurcation point of the self-organized dissipative structure. We obtain the associated eigenvalue problem from Eq. (19) for critical perturbations $\delta\mathbf{u}$ that make $\delta^2 F$ vanish, and the condition for the state with the minimum dissipation rate from Eq. (21), as follows:

$$\nabla \times \nabla \times \delta\mathbf{u}_k - \frac{\alpha_k \rho}{2\nu}\,\delta\mathbf{u}_k = 0\,, \tag{31}$$

$$0 < \alpha < \alpha_1\,. \tag{32}$$

Here $\alpha_k$ is the eigenvalue, $\delta\mathbf{u}_k$ denotes the eigensolution, $\alpha_1$ is the smallest positive eigenvalue, the boundary conditions are $\delta\mathbf{u}_w \cdot d\mathbf{S} = 0$ and $[\,\delta\mathbf{u}_w \times (\nabla \times \delta\mathbf{u}_w)\,] \cdot d\mathbf{S} = 0$, and the subscript $w$ denotes the value at the boundary wall.

Since the present dissipative dynamic operator $L_i^D[\mathbf{q}]$ satisfies the self-adjoint property of Eq. (25) and the boundary conditions are $\delta\mathbf{u}_w \cdot d\mathbf{S} = 0$ and $[\,\delta\mathbf{u}_w \times (\nabla \times \delta\mathbf{u}_w)\,] \cdot d\mathbf{S} = 0$ for Eq. (31), the eigenfunctions $\mathbf{a}_k$ for the associated eigenvalue problem of Eq. (31) form a complete orthogonal set and the appropriate normalization is written as

$$\begin{aligned}\int \mathbf{a}_k \cdot (\nabla \times \nabla \times \mathbf{a}_j)\, dV &= \int \mathbf{a}_j \cdot (\nabla \times \nabla \times \mathbf{a}_k)\, dV \\ &= \frac{\alpha_k \rho}{2\nu} \int \mathbf{a}_j \cdot \mathbf{a}_k\, dV \\ &= \frac{\alpha_k \rho}{2\nu}\,\delta_{jk}\,,\end{aligned} \tag{33}$$

where $\nabla \times \nabla \times \mathbf{a}_k - (\alpha_k\rho/2\nu)\,\mathbf{a}_k = 0$ is used. When the flow-dynamics system has some unstable regions, the nondissipative dynamic operator $L_i^N[\mathbf{q}] = -\nabla p/\rho - (1/2)\nabla u^2 + \mathbf{u} \times \boldsymbol{\omega}$ may become dominant, leading to the rapid growth of perturbations and finally to turbulent phases. This process may yield spectrum transfers toward the larger-mode-number region in the flow $\mathbf{u}$ distribution. The amplitudes of perturbations are considered to grow to nonlinear saturation and not infinitely. We next investigate the change of flow $\mathbf{u}$ distribution for a short time around or after the saturation of perturbation growth. In this phase, operator $L_i^N[\mathbf{q}]$ has become less dominant and $L_i^D[\mathbf{q}]$ becomes more so. The flow $\mathbf{u}$ distribution can be written by using the eigensolution $\mathbf{u}^*$ for the boundary value problem of Eq. (26) for the given boundary value and also by using orthogonal eigenfunctions $\mathbf{a}_k$ for the eigenvalue problem of Eq. (31) with the boundary conditions of $\mathbf{a}_k \cdot d\mathbf{S} = 0$ and $[\,\mathbf{a}_k \times (\nabla \times \mathbf{a}_k)\,] \cdot d\mathbf{S} = 0$ at the boundary, as follows:

$$\mathbf{u} = \mathbf{u}^* + \sum_{k=1}^{\infty} c_k\,\mathbf{a}_k\,. \tag{34}$$

Substituting Eq. (34) into Eq. (23) and using Eqs. (26) and (31), we obtain the following:

$$\frac{\partial \mathbf{u}^*}{\partial t} + \sum_{k=1}^{\infty} \frac{\partial (c_k \mathbf{a}_k)}{\partial t} = L_i^N[\mathbf{q}] - \frac{\alpha}{2}\mathbf{u}^* - \sum_{k=1}^{\infty} \frac{\alpha_k}{2} c_k \mathbf{a}_k\,, \tag{35}$$

where $L_i^N[\mathbf{q}]$ acts now as a less dominant operator, the eigenvalues $\alpha_k$ are positive, and $\alpha_1$ is the smallest positive eigenvalue. We find from Eq. (35) that the flow components of $\mathbf{u}^*$ and $c_k\mathbf{a}_k$ decay approximately by the decay constants of $\alpha/2$ and $\alpha_k/2$, respectively, in the present short time interval, in the same way as in Eqs. (27) and (28). Since the components with the larger eigenvalue $\alpha_k$ decay faster, we see that this decay process yields the higher dissipation rate for the state with the higher-mode-number components. This decay process corresponds to the selective dissipation process demonstrated by three-dimensional MHD simulations in [11–13]. We understand from Eq. (35) that if $\alpha < \alpha_1$, the basic component $\mathbf{u}^*$ remains last and the flow distribution of $\mathbf{u}$ at the minimum dissipation rate phase is represented approximately by $\mathbf{u}^*$. If the value of $\alpha$ becomes greater than $\alpha_1$, then the basic component $\mathbf{u}^*$ decays faster than the eigenmode $\mathbf{a}_1$. This faster decay of the basic component $\mathbf{u}^*$ continues to yield further decrease of $W_{ii}$, resulting in the decrease of $\alpha$ itself, until $\alpha$ becomes equal to $\alpha_1$, i.e., the same decay rate state with the lowest eigenmode $\mathbf{a}_1$. Consequently, the mixed mode which consists of both $\mathbf{u}^*$, having $\alpha = \alpha_1$, and the lowest eigenmode $\mathbf{a}_1$, remains last and the flow distribution of $\mathbf{u}$ at the minimum dissipation rate phase is represented approximately by this mixed mode. The flow energy of this mixed mode decays as $W_{ii}^* \cong e^{-\alpha_1 t} \int (\mathbf{u}_R{}^* + c_1\mathbf{a}_1)^2\, dV$. This argument gives us a detailed physical picture of the self-organization of the dissipative nonlinear dynamic system approaching the basic mode $\mathbf{u}^*$ and also of the bifurcation of the self-organized dissipative structure from the basic mode $\mathbf{u}^*$ to the mixed mode with $\mathbf{u}^*$ and $\mathbf{a}_1$ which takes place at $\alpha = \alpha_1$.

In conclusion, as one of universal mathematical structures embedded in the dissipative dynamic system of Eq. (4), we have shown that the states with the minimum dissipation rate are expressed by the eigenfunctions $q_i^*$ of Eq. (13) for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$. The eigenfunctions $q_i^*$ have been shown to constitute the self-organized and self-similar decay phase of Eq. (15). The factor $\alpha$ of Eq. (13), which is the Lagrangian multiplier, is equal to the decay constant of $W_{ii}$ in the self-organized and self-similar decay phase. The eigenfunctions $q_i^*$ for the dissipative dynamic operators $L_i^D[\mathbf{q}^*]$ are considered to be "the attractor of the dissipative structure." We have presented one typical application of the present theory to the incompressible viscous fluid described by the Navier-Stokes equation of Eq. (22). Using the eigensolution $\mathbf{u}^*$ for the boundary value problem of Eq. (26) for the given boundary value and the complete orthogonal set by the eigenfunctions $\mathbf{a}_k$ for the associated eigenvalue problem of Eq. (31), we have presented a detailed physical picture of the self-organization of the dissipative nonlinear dynamic system approaching the basic mode $\mathbf{u}^*$ and also of the bifurcation of the dissipative structure at $\alpha = \alpha_1$ from the basic mode $\mathbf{u}^*$ to the mixed mode with $\mathbf{u}^*$ and $\mathbf{a}_1$. The helical flow solution of $\nabla \times \mathbf{u} = \kappa \mathbf{u}$, included in the eigensolutions for Eq. (26), is considered to represent approximately the helical flow pattern after the turbulent puffs shown in Fig. 4 of [8]. The present theory is applicable to both resistive MHD plasmas and incompressible viscous MHD fluids [9]. The remarkable point is that the present theory leads to reasonable results [9, 10] even for cases of resistive MHD plasmas with relatively high resistivity where magnetic helicity is no longer time invariant, as shown in Fig. 3 of [12], and for spatially nonuniform resistivity profiles, as shown in Fig. 7 of [14].

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